ON THE DYNAMICS OF THE SELF GRAVITATING ELLIPSOID IN N DIMENSIONS AND ITS DEFORMATION QUANTIZATION

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1. Introduction

The self gravitating ellipsoid has been the subject of study by many physicists and mathematicians. Newton first started the subject in the attempt to determine the eccentricity of the earth, which he modelled using a rigidly rotating ellipsoid made of a fluid of uniform density and subject only to its own gravity. Later on McLaurin generalized and refined his calculation of the eccentricity. It was the first time this number relative to the shape of earth, was calculated with a reasonably accurate model.

The dynamics of such an object was then studied by several people among whom Dedekind, Riemann (often in the literature the self gravitating ellipsoid is referred as the *Riemann ellipsoid*) and more recently Chandrasekhar. For an historical account of the development of this interesting subject see [Le], [Ch].

Recently in [Ro] Rosensteel identified the phase space of the dynamical system associated with the Riemann ellipsoid with the algebra $\mathrm{sym}(3) \times' \mathrm{gl}(3)$, where $\mathrm{sym}(3)$ denotes the symmetric 3×3 matrices and $\mathrm{gl}(3)$ the 3×3 matrices.

A generalization of some of his results was done by Carrero in [Ca]. Carrero introduced an arbitrary number of dimensions and studied in detail the coadjoint action of the group $G = \mathrm{sym}(n) \times' \mathrm{GL}_+(n)$ on its Lie algebra $\mathrm{sym}(3) \times' \mathrm{gl}(3)$. Global Darboux coordinates were explicitly calculated for any coadjoint orbit of G.

Using such global Darboux coordinates one can immediately write a deformation quantization of the orbit using the Moyal Weyl type of deformation quantization. The existence of such a differential deformation, which is unique up to gauge equivalence by Kontsevich's theorem [Kn], does not however guarantee the existence of an algebraic deformation quantization, that is a deformation of the Poisson polynomial algebra of an orbit. The explicit construction of such deformation will be the aim of this paper.

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This paper is organized as follows.

In §2 we study the coadjoint action of $G = \mathrm{sym}(n) \times' \mathrm{GL}_+(n)$, reviewing some of Carrero's results. We then show that there are no invariant polynomials with respect to this action. An explicit description of the regular coadjoint orbits as algebraic varieties is given using semiinvariant polynomials.

In §3 we prove the main result of this paper, namely the existence of a deformation quantization of algebra of polynomial functions of the regular coadjoint orbits of G. The deformation is given explicitly generalizing a construction introduced in [FL1] in the case of a complex semisimple group. The construction of the deformation is non trivial, since the method in [FL1] depends in an essential way on the fact that the group is semisimple, while our G does not have this property.

The existence of an algebraic deformation in this more general setting suggests that a modification of the method in [FL1] could possibly give quantization of more general Poisson algebraic variety. We plan to explore this in a forthcoming paper.

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2. The coadjoint orbits of $G = \mathrm{sym}(n) \times' \mathrm{GL}_+(n)$

Let G be the real Lie group $\mathrm{sym}(n) \times' \mathrm{GL}_+(n)$ with multiplication:

$$(x, g)(y, h) = (x + g^{-1} y \check{g}, gh)$$

where $\mathrm{sym}(n)$ denotes the $n \times n$ real matrices, $\mathrm{GL}_+(n)$ the subgroup of $\mathrm{GL}(n)$ consisting of invertible matrices with positive determinant and $\check{g} = g^{t^{-1}}$.

G can be identified with a subgroup of $\mathrm{Sp}(n)$ in the following way:

$$G = \{(x, g) | x \in \mathrm{sym}(n), g \in \mathrm{GL}_+(n)\} \cong$$

$$\{\begin{pmatrix} g & x\check{g} \\ 0 & \check{g} \end{pmatrix} | x \in \mathrm{sym}(n), g \in \mathrm{GL}_+(n)\} \subset \mathrm{Sp}(n)$$

It is immediate to check that with such identification we have that:

$$\mathcal{G} = Lie(G) \cong \{\begin{pmatrix} a & b \\ 0 & -a^t \end{pmatrix} | a \in \mathrm{gl}(n), b \in \mathrm{sym}(n)\} \subset \mathrm{sp}(n)$$

Let's consider the non degenerate form on $\mathrm{sp}(n)$:

$$< A, B >= \mathrm{tr}(AB).$$

This form is still non degenerate on $\mathcal{G} \times \mathcal{G}_- \subset \mathrm{sp}(n) \times \mathrm{sp}(n)$, where

$$\mathcal{G} = \{ \begin{pmatrix} a & b \\ 0 & -a^t \end{pmatrix} | a \in \mathrm{gl}(n), b \in \mathrm{sym}(n) \}$$

$$\mathcal{G}_- = \{ \begin{pmatrix} a & 0 \\ c & -a^t \end{pmatrix} | a \in \mathrm{gl}(n), c \in \mathrm{sym}(n) \}$$

This allows us to identify $\mathcal{G}^* \cong \mathcal{G}_-$.

For brevity we will denote:

$$\begin{pmatrix} a & b \\ 0 & -a^t \end{pmatrix} \in \mathcal{G}^* \quad \text{with} \quad (b,a), \qquad \begin{pmatrix} a & 0 \\ c & -a^t \end{pmatrix} \in {\mathcal{G}_-}^* \quad \text{with} \quad (c,a)$$

and

$$\begin{pmatrix} g & x\check{g} \\ 0 & \check{g} \end{pmatrix} \in G \quad \text{with} \quad (x,g).$$

In the above notation we have that the adjoint and coadjoint actions of G on $\mathcal{G}$ and $\mathcal{G}^*$ respectively are given by:

$$Ad(x,g)(b,a) = (gbg^t - \{gag^{-1}x + (gag^{-1}x)^t\}, gag^{-1})$$

$$Ad^*(x,g)(c,a) = (\check{g}cg^{-1}, gag^{-1} + x\check{g}cg^{-1})$$

Define

$$\mathcal{G}^+ = \{(c,a) \in \mathcal{G}^* | c \text{ positive definite}\}$$

We are now interested in the description of the coadjoint orbits of G of elements in $\mathcal{G}^+$. These are the orbits physically interesting.

Notice that $\mathcal{G}^+$ is an open set in $\mathcal{G}$ invariant under the coajoint action. Let $O_{(c,a)}$ denote the coadjoint orbit of an element (c,a). Moreover one can immediately see that:

$$O_{(c,a)} = O_{(I,d)}$$

where I is the identity matrix and $d \in \mathrm{gl}(n)$.

Lemma (2.1). *Let $(c,a) \in \mathcal{G}^+$ and let $O_{(c,a)}$ be the coadjoint orbit of (c,a).*

1) If $n = 2k+1$, there exists a unique element (I,H) such that $O_{(c,a)} = O_{(I,H)}$ with

$$H = \begin{pmatrix} 0 & -\lambda_1 & \dots & & & \\ -\lambda_1 & 0 & & & & \\ & & \vdots & & & \\ & & & 0 & \lambda_k & \\ & & & -\lambda_k & 0 & \\ & & & & & 0 \end{pmatrix}$$

and $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k$.

2) If $n = 2k$ *there exists a unique element* (I, H) *such that* $O_{(c,a)} = O_{(I,H)}$ *with*

$$H = \begin{pmatrix} 0 & -\lambda_1 & \ldots & & & \\ -\lambda_1 & 0 & & & & \\ & & & & & \\ & & \vdots & & & \\ & & & & & \\ & & & & 0 & \lambda_k \\ & & & & -\lambda_k & 0 \end{pmatrix}$$

and $\lambda_1 \geq \lambda_2 \geq \ldots \geq l_k$.

3) If $\lambda_1 > \lambda_2 > \ldots > \lambda_k$,

$$O_{(c,a)} = G/\mathrm{SO}(2) \times \ldots \times \mathrm{SO}(2)$$

hence dim $O_{(c,a)} = n^2 - k$.

Proof. See [Ca].

Let $\mathcal{H}$ be the Cartan subalgebra of so(n) defined in Lemma (2.1). Let $\mathcal{I}(\mathcal{H})$ be the algebra of invariant polynomials under the action of the Weyl group $\mathcal{W}$ of so(n). We know that this algebra is the same as the algebra of polynomials on so(n) invariant under the adjoint action. Since every G orbit in $\mathcal{G}^+$ meets $\mathcal{H}$ in a $\mathcal{W}$ orbit we have that an invariant function on $\mathcal{G}^+$ is determined uniquely by its restriction to $\mathcal{H}$. Let $\mathcal{I}(\mathcal{G}^+)$ be the algebra of invariant functions on $\mathcal{G}^+$. We have the following result.

Theorem (2.2). *1) If* $n = 2k$ *then* $\mathcal{I}(\mathcal{H})$ *is generated by:*

$$\tilde{f}_i = \textstyle\sum_{j=1}^k \lambda_j^{2i} \quad i = 1 \ldots k-1, \quad \tilde{f}_k = \lambda_1 \ldots \lambda_k$$

and $\mathcal{I}(\mathcal{G}^+)$ *is generated by:*

$$f_i = \tfrac{1}{2^{2i}} \mathrm{tr}(cac^{-1} - a^t)^{2i}, \quad i = 1 \ldots k-1, \quad \mathrm{Pf} = \pi(ac^{-1}) det(c)^{1/2}$$

with $\pi(A)$ *denoting the Pfaffian of the matrix* $1/2(A - A^t)$.

2) If $n = 2k+1$, *then* $\mathcal{I}(\mathcal{H})$ *is generated by:*

$$\tilde{f}_i = \textstyle\sum_{j=1}^k \lambda_j^{2i} \quad i = 1 \ldots k$$

and $\mathcal{I}(\mathcal{G}^+)$ *is generated by:*

$$f_i = \tfrac{1}{2^{2i}} \mathrm{tr}(cac^{-1} - a^t)^{2i}, \quad i = 1 \ldots k.$$

Proof. See [Ca].

Notice that there are rational and irrational functions in $\mathcal{I}(\mathcal{G}^+)$ that are not polynomial in (c, a).

We now want to determine the subring of invariant polynomials $\mathcal{IP}(\mathcal{G}^+) \subset \mathcal{I}(\mathcal{G}^+)$.

Proposition (2.3). *If $n = 2k$, the ring $\mathcal{IP}(\mathcal{G}^+)$ of invariant polynomials on ${\mathcal{G}^+}^*$ consists only of constants.*

Proof. We first observe that

$$\mathcal{I}(\mathcal{G}^+) = span_{l\in\mathbf{Z}}\mathrm{tr}(cac^{-1} - a^t)^{2l}$$

Assume that the polynomial $f(a, c) \in \mathcal{I}(\mathcal{G}^+)$. This means that

$$f(c, a) = \sum_{\lambda_l\in\mathbf{C}} \lambda_l \mathrm{tr}(cac^{-1} - a^t)^{2l}$$

Let $-2M$ be the highest negative degree of $det(c)$. Then we have the equation between polynomials:

$$det(c)^{2M} f(c, a) = \sum_{\lambda_l\in\mathbf{C}} \lambda_l det(c)^{2M-2l}\mathrm{tr}(caC^t - a^t)^{2l}$$

where C is the matrix of the algebraic complements.

Observe that an invariant polynomial must depend on both a and c. We will prove that it depends only on a reaching a contradiction. Let $\deg_{ij}$ denote the degree in c_{ij} of a generic polynomial, where c_{ij} is the (i, j) entry of the matrix c.

Claim. $\deg_{ij}(\mathrm{tr}(caC^t - a^t)^{2l} \leq 2l$.

Given a matrix x let's associate to it another matrix $(x)_{\deg_{ij}}$ whose (k, l) entry is $\deg_{ij}(x_{kl})$.

One can easily see the following:

$$(c)_{\deg_{ij}} = E_{ij}$$

$$(a)_{\deg_{ij}} = \mathbf{0}$$

$$(C^t)_{\deg_{ij}} = \textstyle\sum_{1\leq r,s\leq k, r\neq i, s\neq j} E_{rs}$$

where E_{ij} denotes the elementary $k \times k$ matrix having 1 in the (i, j) position and 0 everywhere else and $\mathbf{0}$ denotes the $k \times k$ zero matrix.

By induction one gets:

$$((caC^t)^m)_{\deg_{ij}} = \sum_{1\leq r,s\leq k, r\neq i, s\neq j} mE_{rs} + \sum_{1\leq r\leq k, r\neq i} (m-1)E_{rj} + \sum_{1\leq s\leq k, s\neq j} (m+1)E_{rs} + mE_{ij}$$

From which we have $\deg_{ij}(caC^t)^{2l} \leq 2l+1$, $i \neq j$, $\deg_{ii}(caC^t)^{2l} \leq 2l$.

Now we compute $\deg_{ij}(f(c,a))$. By the claim we have

$$\deg_{ij}(\sum_{\lambda_l \in \mathbf{C}} \lambda_l det(c)^{2M-2l} \mathrm{tr}(caC^t - a^t)^{2l}) \leq \begin{cases} 2M+1 & \text{if } i \neq j \\ 2M & \text{if } i = j \end{cases} .$$

But observe that

$$\deg_{ij}(det(c)^{2M}) = 2M.$$

Hence $\deg_{ij}(f(c,a)) \leq 1$ if $i \neq j$, $\deg_{ii}(f(c,a)) = 0$. This implies that $f(c,a)$ does not depend on c_{ii}. Now assume

$$f(c,a) = \sum_{b_{ij} \in \mathbf{C}[a], i \neq j} b_{ij} c_{ij}.$$

where $\mathbf{C}[a]$ denotes the ring of polynomials in the indeterminates a_{ij}'s. Now choose $(x,g) \in G$ such that $\check{g}_{j_0 i_0} \neq 0$, $g^{-1}_{j_0 j_0} \neq 0$. It is a simple computation to check that

$$(\check{g} c \check{g}^{-1})_{j_0 j_0} \neq 0.$$

This implies that $f(c,a)$ must also depend on $c_{j_0 j_0}$ which is a contradiction. So $f(c,a)$ must depend only on a, but this is not possible since it is invariant, unless it is a constant. QED.

We now want to ask whether there are semiinvariants for the coadjoint action.

Define the polynomials:

Odd case:

$$h_i = det(c)^{2i} \mathrm{tr}(cac^{-1} - a^t)^{2i} \quad i = 1 \dots k$$

Even case:

$$h_i = det(c)^{2i} \mathrm{tr}(cac^{-1} - a^t)^{2i} \quad i = 1 \dots k-1,$$

$$h_k = det(c)^2 \pi(ac^{-1})^2 det(c)$$

where $\pi(ac^{-1}) = \mathrm{Pf}(1/2(ac^{-1} - (ac^{-1})^t)$ and Pf denotes the Pfaffian.

One can easily check that these algebraic functions on $\mathcal{G}^*$: are semiinvariant for the coadjoint action. In fact:

$$Ad^*_{(x,g)} h_m = det(g)^{-4m} h_m$$

It would be interesting to determine all semiinvariants.

We now would like to describe the coadjoint orbits as algebraic varieties. For this reason is now more convenient to look at their complexification. Let $O^{\mathbf{C}}_{(c,a)}$ denote the complexification of the orbit $O_{(c,a)}$.

It is clear that we cannot describe the ideal of the orbit in the same way as in the semisimple case. In fact in that case we have that the ideal of a given regular orbit is simply given by the polynomials that Chevalley generators of the ring of invariant polynomials equal to constant ([Ko]). We will need to use the semiinvariant polynomials.

Theorem (2.4). *Given a regular orbit $O_{(c,a)}$, its ideal is given by:*

1) If $n = 2k + 1$, $(h_1 - \alpha_1 det(c)^2, \dots h_k - \alpha_k det(c)^{2k})$

2) If $n = 2k$, $(h_1 - \alpha_1 det(c)^2, \dots h_k - \alpha_k det(c)^2)$

where α_i for $i = 1 \dots k-1$ is the constant value of the rational function $\mathrm{tr}(cac^{-1} - a^t)^{2i}$ on the orbit. α_k is the the constant value of the rational function $\mathrm{tr}(cac^{-1} - a^t)^{2k}$ on the orbit if $n = 2k + 1$, while if $n = 2k$ it is the constant value of $\pi(ac^{-1})^2 det(c)$ on the orbit.

Proof. Let $r_1 \dots r_k$ be the generators of the ideal. Since the orbit is a non singular algebraic variety of dimension $n^2 - k$ it is enough to prove that the differentials $dr_1 \dots dr_k$ are linearly independent over every point of $O_{(c,a)}$. Since there is a diffeomorphism that brings any point of $O_{(c,a)}$ into any other point, it is enough to prove the differentials are linearly independent over points in $\mathcal{H}$.

Observe that $d(r_1|_{\mathcal{H}}), \dots d(r_k|_{\mathcal{H}})$ are linearly independent over every regular point of $\mathcal{H}$ ([Va2]).

Hence it is simple to see that also $(dr_1)|_{\mathcal{H}} \dots (dr_k)|_{\mathcal{H}}$ are linearly independent over every regular point of $\mathcal{H}$. Q.E.D.

3. Deformation quantization of regular coadjoint orbits

We would like to construct a deformation quantization of the algebra of regular functions on a regular coadjoint orbit of a Lie group under certain hypothesis listed below, which are satisfied by $G = \mathrm{sym}(n) \times' \mathrm{GL}_+(n)$. Our construction is a generalization of the one described in [FL1] where G was assumed to be complex semisimple.

Let's recall the basic definitions.

Definition (3.1). Given a real (or complex) Poisson algebra $\mathbf{P}$, a *formal deformation* or a *deformation quantization* of $\mathbf{P}$ is an associative algebra $\mathbf{P}_h$ over $\mathbf{R}[[h]]$ (or over $\mathbf{C}$[[h]]), where h is a formal parameter, with the following properties:

a. $\mathbf{P}_h$ is isomorphic to $\mathbf{P}[[h]]$ as a $\mathbf{R}[[h]]$-module (or as a $\mathbf{C}[[h]]$-module).

b. The multiplication $*_h$ in $\mathbf{P}_h$ reduces mod(h) to the one in $\mathbf{P}$.

c. $\tilde{F} *_h \tilde{G} - \tilde{G} *_h \tilde{F} = h\{F, G\}$ mod (h^2), where $\tilde{F}, \tilde{G} \in \mathbf{P}_h$ reduce to $F, G \in \mathbf{P}$ mod(h) and $\{\,,\}$ is the Poisson bracket in $\mathbf{P}$.

This definition makes sense also if we substitute $\mathbf{C}[[h]]$ by $\mathbf{C}[h]$. In this case we will say that P_h is a $\mathbf{C}[h]$-*deformation*.

Notice that a $\mathbf{C}[h]$-deformation extends immediately to a formal one by tensoring by $\mathbf{C}[[h]]$, but the converse is not always true. Moreover $\mathbf{C}[h]$-deformation can be specialized to any value of the parameter h.

Let's now make the following assumptions.

1. $\mathcal{G}$ is a finite dimensional complex Lie algebra of a complex Lie group G.

2. $p_1 \dots p_m \in \mathbf{C}[\mathcal{G}^*]$ are semiinvariant polynomials with respect to the coadjoint action.

3. $dp_1 \dots dp_m$ are linearly independent over points where $p_1 = \dots = p_m = 0$.

4. The set of zeros of $p_1 \dots p_m$ is an algebraic Poisson variety with bracket induced by the one in $\mathcal{G}^*$.

In these hypothesis we will construct a deformation quantization of the algebraic Poisson variety described by the ideal $(p_1 \dots p_m) \subset \mathbf{C}[\mathcal{G}^*]$, i.e. a formal deformation of the Poisson algebra $\mathbf{C}[\mathcal{G}^*]/(p_1 \dots p_m)$.

Observation (3.2). Notice that if we take $\mathcal{G} = Lie(\mathrm{sym}(n) \times' \mathrm{GL}_+(n))$, $m = k$ and $p_i = h_i - \alpha_i det(c)^{2i}$, $i = 1 \dots k-1$, and if $n = 2k+1$ $p_k = h_k - \alpha_i det(c)^{2k}$, if $n = 2k$ $p_k = h_k - \alpha_i det(c)^2$ (for the notation see §2) the hypothesis (1), (2), (3), (4) listed above are satisfied. Hence the procedure described below will give us a deformation quantization of the algebra of polynomial function on a regular coadjoint orbit of $\mathrm{sym}(n) \times' \mathrm{GL}_+(n)$, $O^{\mathbf{C}}_{(c,a)}$, set of zeros of such $p_1 \dots p_k$.

Let's denote by $T_A(V)$ the full tensor algebra of a complex vector space V over a $\mathbf{C}$-algebra A. Let $\mathcal{G} = Lie(G)$. Consider the proper two sided ideal in $T_{\mathbf{C}[h]}(\mathcal{G})$

$$\mathcal{L}_h = \sum_{X,Y \in \mathcal{G}} T_{\mathbf{C}[h]}(\mathcal{G}) \otimes (X \otimes Y - Y \otimes X - h[X,Y]) \otimes T_{\mathbf{C}[h]}(\mathcal{G})$$

U_h is a free $\mathbf{C}[[h]]$-module, in particular it is torsion free ([FL1] Proposition (3.2)).

Define $U_h =_{def} T_{\mathbf{C}[h]}(\mathcal{G})/\lambda_h$. U_h is the universal enveloping algebra of the Lie algebra $\mathcal{G}_h = \mathbf{C}[h] \otimes_{\mathbf{C}} \mathcal{G}$ with Lie bracket

$$[p(h)X, q(h)Y]_h = p(h)q(h)[X,Y]$$

Let $\mathbf{C}[X]$ denote the regular (polynomial) functions on a variety X. Define $\mathbf{C}_h[\mathcal{G}^*] = \mathbf{C}[h] \otimes \mathbf{C}[\mathcal{G}^*]$. Observe that $\mathbf{C}_h[\mathcal{G}^*] \cong \mathbf{C}[\mathcal{G}^*_h]$. Let Sym denote the symmetrizer map, $\mathrm{Sym} : \mathbf{C}_h[\mathcal{G}^*] \longrightarrow U_h$ ([Va1] pg. 180).

Denote also with I_h the two-sided ideal in U_h generated by $P_1 = \mathrm{Sym}(p_1), \dots P_m = \mathrm{Sym}(p_m)$.

Proposition (3.2). *If $p \in \mathbf{C}[\mathcal{G}^*]$ is semiinvariant with respect to the coadjoint action i.e.: $Ad^*(g)p = f(g)p$ for all $g \in G$, where f is a function depending only on g, then*

$$[X, \mathrm{Sym}(p)] = F(X)\mathrm{Sym}(p) \qquad \textit{for all } X \in U_h,$$

with F scalar function depending only on X.

Proof. Direct calculation.

Lemma (3.3). *Let r be a fixed positive integer and let all the notation be as above. Let*

$$\sum_{1 \leq i_1 \leq \cdots \leq i_r \leq m} a_{i_1 \dots i_r} p_{i_1} \dots p_{i_r} = 0$$

with $a_{i_1 \dots i_r} \in \mathbf{C}[\mathcal{G}^*]$. Then $a_{i_1 \dots i_r} \in (p_1, \dots, p_m) \subset \mathbf{C}[\mathcal{G}^*]$.

Proof. The proof is the same as in Proposition (3.8), [FL1].

Lemma (3.4). *Let k be a fixed integer and let*

$$\sum_{i_1 \leq \cdots i_k \leq m} A_{i_1 \dots i_k} P_{i_1} \cdots P_{i_k} \equiv 0 \qquad \textit{mod} h$$

where $A_{i_1 \dots i_k} \in U_h$ and $P_i = \mathrm{Sym}(p_i)$.

Then

$$\textstyle\sum_{i_1 \leq \cdots i_k \leq m} A_{i_1 \dots i_k} P_{i_1} \cdots P_{i_k} = h \sum_{i_1 \leq \cdots i_k \leq m} B_{j_1 \dots j_l, i_1 \dots i_k}$$

$$P_{j_1} \cdots P_{j_l} P_{i_1} \cdots P_{i_k}$$

Proof. This is the same as Lemma (3.9), [FL1].

Lemma (3.5). *If $hF \in I_h$ then $F \in I_h$. In other words, U_h/I_h is torsion free.*

Proof. Since $hF \in I_h$ and since Proposition (3.2) we can write:

$$hF = \sum A_i P_i$$

We have $\sum A_i P_i \equiv 0 \bmod h$. Hence, by Lemma (3.4) and also by the fact that U_h is torsion free we have our result.

We now want to construct a basis for the torsion free $\mathbf{C}[h]$-module U_h/I_h.

Let's fix a basis $\{X_1, \dots, X_n\}$ of $\mathcal{G}$ and let $x_1, \dots, x_n$ be the corresponding elements in $\mathbf{C}[\mathcal{G}^*]$. With this choice $\mathbf{C}[\mathcal{G}^*] \cong \mathbf{C}[x_1, \dots, x_n]$. Let $\{x_{i_1}, \dots, x_{i_k}\}_{(i_1, \dots, i_k) \in \mathcal{A}}$ be a basis in of $\mathbf{C}[\mathcal{G}^*]/(p_1 \dots p_m)$ as $\mathbf{C}$-module, where $\mathcal{A}$ is a set of multiindices appropriate to describe the basis. In particular, we can take them such that $i_1 \leq \cdots \leq i_k$.

Proposition (3.6). *The monomials $\{X_{i_1} \cdots X_{i_k}\}_{(i_1, \dots, i_k) \in \mathcal{A}}$ are a basis for U_h/I_h.*

Proof. The proof is exactly the same as the one in Proposition (3.11) and (3.13) in [FL1].

Theorem (3.7). *Let the notation be as above.*

1. U_h/I_h *is a* $\mathbf{C}[h]$*-deformation of* $\mathbf{C}[\mathcal{G}^*]/(p_1 \dots p_m)$*.*

2. $U_h/I_h \otimes \mathbf{C}[[h]]$ *is a deformation quantization of* $\mathbf{C}[\mathcal{G}^*]/(p_1 \dots p_m)$*.*

Proof. Immediate from previous lemmas.

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